Effective Descriptive Set Theory and Applications in Analysis

Vassilis Gregoriades

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Suslin's Theorem.

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Recursion Theory on ω^k.

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Parametrization. For every $\varepsilon \in \mathcal{N} = \omega^\omega$ one defines the *relativized* family $\mathcal{A}(\varepsilon)$ of *ε-recursive* functions.

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A set $P \subseteq \omega^k$ is *recursive* when the characteristic function χ_p is recursive.

P *ε-recursive* when χ_p is ε-recursive.

Effective Theory.

Suppose that $\mathcal{X}$ is a Polish space, d is compatible distance function for $\mathcal{X}$ and $(x_n)_{n\in\omega}$ is a sequence in $\mathcal{X}$. Define the relation $P_<$ of ω^4 as follows $P_<(i,j,k,m) \iff d(x_i,x_j) < \frac{k}{m+1}$. Similarly we define the relation $P_\leq$.

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Without loss of generality we will deal with recursively presented Polish spaces.

$N(\mathcal{X}, s) =$ the ball with center $x_{(s)_0}$ and radius $\frac{(s)_1}{(s)_2+1}$.

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$\rightsquigarrow$ effective open sets.

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Similarly one defines the *relativized* pointclasses with respect to some parameter ε which will be denoted by $\Delta^1_1(\varepsilon)$ for instance.

A function $f : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if the set $R^f \subseteq \mathcal{X} \times \omega$, $R^f(x, s) \iff f(x) \in N(\mathcal{Y}, s)$ is open.

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A point $x \in \mathcal{X}$ is Δ^1_1 *point* if the relation $U \subseteq \omega$ which is defined by

$$U(s) \iff x \in N(\mathcal{X}, s)$$

is Δ^1_1.

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Effective Theory on Polish spaces has been developed with the following preassumption: every recursively presented Polish space which is not of the form ω^k is perfect i.e., there are no isolated points. But on the other hand all previous notions are given in general.

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Theorem (VG). Let $\mathcal{X}$ be an uncountable recursively presented Polish space. The following are equivalent.

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Corollary (VG). $\Sigma^1_1 \neq \Pi^1_1$.

Some Questions.

Let $\mathcal{X}$, $\mathcal{Y}$ be two recursively presented Polish spaces. Define $\mathcal{X} \preccurlyeq \mathcal{Y}$ if there is a Δ^1_1 *injection* $\pi : \mathcal{X} \to \mathcal{Y}$.
Define also $\mathcal{X} \approx \mathcal{Y}$ if there is a Δ^1_1 *isomorphism* $\pi : \mathcal{X} \to \mathcal{Y}$.

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For every recursively presented Polish space $\mathcal{X}$ we have that

$$\omega \preccurlyeq \mathcal{X} \preccurlyeq \mathcal{N}.$$

Of course if $\mathcal{X}$ is uncountable then $\omega \prec \mathcal{X}$, in particular $\omega \prec \mathcal{N}$.

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Fix a recursive tree T such that the body $[T]$ does not contain any Δ^1_1 members. Then we have the following

$$\omega \prec \mathcal{X}^T \prec \mathcal{N}.$$

Every recursively presented Polish space $\mathcal{X}$ is Δ^1_1 isomorphic to a space of the form $\mathcal{X}^T$ for some recursive tree on ω.

If $T = \omega^{<\omega}$ then $\mathcal{X}^T \approx \mathcal{N}$.

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$$T_1 \sim T_2 \Longleftrightarrow \mathcal{X}^{T_1} \approx \mathcal{X}^{T_2}.$$

Can we express the equivalence relation $\sim$ in terms of the *combinatorial* and the *effective* properties of the trees T_1, T_2?

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Question no5. Is the relation $\preccurlyeq$ linear?

Connections with General Topology.

Theorem (Bourgain-Fremlin-Talagrand). Let $\mathcal{X}$ be a Polish space and $(f_n)_{n\in\omega}$ be a sequence of Borel-measurable functions from $\mathcal{X}$ to $\mathbb{R}$ which satisfies (1) the sequence $(f_n)_{n\in\omega}$ is pointwise bounded and (2) every cluster point of $(f_n)_{n\in\omega}$ in $\mathbb{R}^{\mathcal{X}}$ with the product topology is a Borel-measurable function. Then there is a subsequence $(f_n)_{n\in L}$ which is pointwise convergent.

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Theorem (Debs). Let $\mathcal{X}$ be a recursively presented Polish space and $(f_n)_{n\in\omega}$ be a sequence of continuous functions from $\mathcal{X}$ to $\mathbb{R}$ which satisfies conditions (1) and (2) above and in addition (3) the sequence $(f_n)_{n\in\omega}$ is $\Delta^1_1(\alpha)$-recursive.

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Theorem (Debs). Let $\mathcal{X}$ be a recursively presented Polish space and $(f_n)_{n\in\omega}$ be a sequence of continuous functions from $\mathcal{X}$ to $\mathbb{R}$ which satisfies conditions (1) and (2) above and in addition (3) the sequence $(f_n)_{n\in\omega}$ is $\Delta^1_1(\alpha)$-recursive. Then there is an infinite $L \subseteq \omega$ which is in $\Delta^1_1(\alpha)$ such that the subsequence $(f_n)_{n\in L}$ is pointwise convergent.

Theorem (Extension of the Topology). Suppose $(\mathcal{X}, \mathcal{T})$ is a Polish space and A is a Borel subset of $\mathcal{X}$. Then there is a Polish topology $\mathcal{T}_A$ which extends $\mathcal{T}$, the set A is $\mathcal{T}_A$-clopen and the topologies $\mathcal{T}_A$ and $\mathcal{T}$ have the same Borel sets.

From this it follows that if $(f_n)_{n\in\omega}$ is a sequence of Borel-measurable functions from $(\mathcal{X}, \mathcal{T})$ to $\mathbb{R}$ then there exists a Polish topology $\mathcal{T}'$ which extends $\mathcal{T}$, has the same Borel sets with $\mathcal{T}$ and every function f_n is $\mathcal{T}'$-continuous.

Connections with General Topology.

Theorem (Bourgain-Fremlin-Talagrand). Let $\mathcal{X}$ be a Polish space and $(f_n)_{n\in\omega}$ be a sequence of Borel-measurable functions from $\mathcal{X}$ to $\mathbb{R}$ which satisfies (1) the sequence $(f_n)_{n\in\omega}$ is pointwise bounded and (2) every cluster point of $(f_n)_{n\in\omega}$ in $\mathbb{R}^{\mathcal{X}}$ with the product topology is a Borel-measurable function. Then there is a subsequence $(f_n)_{n\in L}$ which is pointwise convergent.

Theorem (Debs). Let $\mathcal{X}$ be a recursively presented Polish space and $(f_n)_{n\in\omega}$ be a sequence of continuous functions from $\mathcal{X}$ to $\mathbb{R}$ which satisfies conditions (1) and (2) above and in addition (3) the sequence $(f_n)_{n\in\omega}$ is $\Delta^1_1(\alpha)$-recursive. Then there is an infinite $L \subseteq \omega$ which is in $\Delta^1_1(\alpha)$ such that the subsequence $(f_n)_{n\in L}$ is pointwise convergent.

Theorem (Debs, 2009). Suppose $\mathcal{X}$ is a recursively presented Polish space and $(f_n)_{n\in\omega}$ is a sequence of functions from $\mathcal{X}$ to $\mathbb{R}$ which is $\Delta^1_1(\alpha)$-recursive (and so it consists of Borel-measurable functions) with the following properties:

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(3) a set $B \subseteq \mathcal{X}$ is a $\Delta^1_1(\varepsilon, \alpha)$ subset of $(\mathcal{X}, \mathcal{T})$ if and only if B is a $\Delta^1_1(\varepsilon, \alpha)$ subset of $(\mathcal{X}, \mathcal{T}_A)$.

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Theorem (*VG*). Suppose $\mathcal{X}$ is a recursively presented Polish space and $(f_n)_{n\in\omega}$ is a sequence of functions from $\mathcal{X}$ to $\mathbb{R}$ which is $\Delta^1_1(\alpha)$-recursive (and so it consists of Borel-measurable functions) with the following properties:

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Then there is an $\varepsilon \in \mathcal{N}$ and $L \in \Delta^1_1(\varepsilon, \alpha)$ such that the subsequence $(f_n)_{n\in L}$ is pointwise convergent and moreover ε is the characteristic function of a Σ^1_1 subset of ω.

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Results in Banach space Theory.

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Theorem (VG). Let X be a separable Banach space. Then the set

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Let Q be a coanalytic subset of $X^\omega \times X$. Then the set

$$\begin{aligned} P_Q \quad = \quad & \{\, (y_i)_{i\in\omega} \in X^\omega \ /\ \text{the sequence } (y_i)_{i\in\omega} \text{ is weakly convergent} \\ & \text{to some } y \text{ and } Q((y_i)_{i\in\omega}, y) \,\} \end{aligned}$$

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Theorem (Erdös-Magidor). Let X be a Banach space and $(e_i)_{i\in\omega}$ be a bounded sequence in X. Then there is a subsequence $(e_{k_i})_{i\in\omega}$ such that: either (I) every subsequence of $(e_{k_i})_{i\in\omega}$ is Cesàro summable with respect to the norm and all being summed to the same limit; or (II) *no* subsequence of $(e_{k_i})_{i\in\omega}$ is Cesàro summable.

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Theorem (VG). Let X be a Banach space, $(e_i)_{i\in\omega}$ be a bounded sequence in X and let $Q \subseteq X^\omega \times X$ be a coanalytic set. Then there is a subsequence $(e_i)_{i\in L}$ of $(e_i)_{i\in\omega}$ for which: either
(I) there is some $e \in X$ such that every subsequence $(e_i)_{i\in H}$ of $(e_i)_{i\in L}$ is weakly Cesàro summable to e and $Q((e_i)_{i\in H}, e)$; or
(II) for every subsequence $(e_i)_{i\in H}$ of $(e_i)_{i\in L}$ and every $e \in X$ with $Q((e_i)_{i\in H}, e)$ the sequence $(e_i)_{i\in H}$ is not weakly Cesàro summable to e.

Example. Let $(f_n)_{n\in\omega}$ be a bounded sequence of differentiable functions. Then there is a subsequence $(f_n)_{n\in L}$ such that: either (I) there is a differentiable function f such that for every $H \subseteq L$ the sequences $(f_n)_{n\in H}$ and $(f'_n)_{n\in H}$ are pointwise Cesàro summable to f and f' respectively; or (II) for every differentiable function f and every $H \subseteq L$ if $(f_n)_{n\in H}$ is pointwise Cesàro summable to f then then $(f'_n)_{n\in H}$ is not pointwise Cesàro summable to f'.

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Theorem (VG). Let X be a Banach space and $(e_i)_{i\in\omega}$ a bounded sequence in X for which every subsequence $(e_i)_{i\in L}$ has a further subsequence $(e_i)_{i\in H}$ which is weakly Cesàro summable. Then

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(1) every subsequence of $(e_i)_{i\in\omega}$ has a weakly *convergent* subsequence and

(2) there is a Borel-measurable function $f : [\mathbb{N}]^\omega \to [\mathbb{N}]^\omega$ such that for all subsequences $(e_i)_{i\in L}$ the sequence $(e_i)_{i\in f(L)}$ is a weakly convergent subsequence of $(e_i)_{i\in L}$.

Danke schön!